Markov Chain Monte Carlo for Rare Event Reliability Analysis with Nonlinear Finite Elements

Uncertainty Quantification 2016 - MS75 Part II of II
Theory and Simulation of Failure Probabilities and Rare Events

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Overview

Question: Which MCMC methodology is most appropriate for Subset Simulation when sampling is computationally expensive?

Test case: a footing on soil with a nonlinear material model

Contents:

- Briefly: Why rare event estimation is important in Civil Engineering
- Subset Simulation and Markov Chain Monte Carlo for FEM
- Numerical problem description
- Results and computational efficiency comparison

Rare events: relevance and simulation challenges

Risk in Civil Engineering:

Unacceptable performance carries severe consequences. Loss of life and large damage cost possible during failures. This necessitates that the probability of unacceptable performance is small.

Rare event simulation for stochastic PDE based reliability analysis:

Series expansion methods	Monte Carlo Simulation
- Faster than sampling for close to mean responses. - Poor accuracy far from mean. - Nonlinear material models are a challenge.	- Fixed convergence rate $(\frac{1}{\sqrt{N}})$ with number of simulations, N. - Slow for far from mean response. - Nonlinear material models only as challenging in the deterministic case.

Subset Simulation

Denote the threshold event of interest as T.

Decompose the probability of T occurring as a series of conditional probabilities:

$$\mathbb{P}(T) = \mathbb{P}(T_1) \prod_{i}^{m-1} \mathbb{P}(T_{i+1}|T_i)$$

where:

$$T_1 \subset T_2 \subset \cdots \subset T_m = T$$

Estimate $\mathbb{P}(T_{i+1}|T_i)$ by Markov Chain Monte Carlo.

If sampled value $\notin T_i$, reject the sample and revert to previous sample value.

Markov Chain Monte Carlo for Subset Simulation

As in regular Monte Carlo, estimate probabilities by sample estimate. For N FEM simulations:

$$\mathbb{P}(T_{i+1}|T_i) = \frac{1}{N}\sum_{j=1}^{N}\Psi(u_j)$$

where u_j is solution to j-th FEM simulation and $\Psi(u_j) = 1$ if u_j has T_{i+1} occur and $\Psi(u_j) = 0$ otherwise.

Monte Carlo Simulation draws the next sample from the input distribution independently of the previous sample.

Markov Chain Monte Carlo finds next sample by applying a transition function to the current sample.

Error estimation - stationary Markov Chains

Central Limit Theorem for MCS and stationary Markov Chains:

$$\mathcal{N}\left(\mu, \frac{\sigma^2}{N}\right)$$

For stationary Markov Chains - MCMC mean estimate variance:

$$\sigma^2 = \mathrm{Var}\left[\Psi(u)\right] + 2\sum_{k=1}^{\infty} \mathrm{Cov}\left[\Psi(u_j), \Psi(u_{j+k})\right]$$

Where $\Psi(u_j)$ indicates $u_j \in T_{i+1}$. Let $\Psi(u)$ represent $\Psi(u_j)$ for all j.

Subset Simulation product probability distribution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y\left(\frac{z}{x}\right) f_X(x) \frac{1}{|x|} dx$$

Can use this equation to numerically estimate confidence intervals.

MCMC Sampling methodology summary

Metropolis-Hastings Ratio - Probability to update random walk sample

Acceptance ratio: $\alpha = \frac{\mathbb{P}(\text{new})}{\mathbb{P}(\text{old})}$. Accept new sample with probability α.

Represent input sample as vector in $\mathbb{R}^D$

Metropolis-Hastings (MH)	Gibbs Sampling	Componentwise-Metropolis-Hastings (CMH)
• Update entire vector • Use transition function at previous result • Accept new value according to ratio	• Update single component • Update according to marginal distribution • Always accept new value	• Update single component • Use transition function at previous result • Accept new value according to ratio

MCMC Sampler efficiency for FEM analysis

The Subset Simulation MCMC sampler efficiency will depend on:

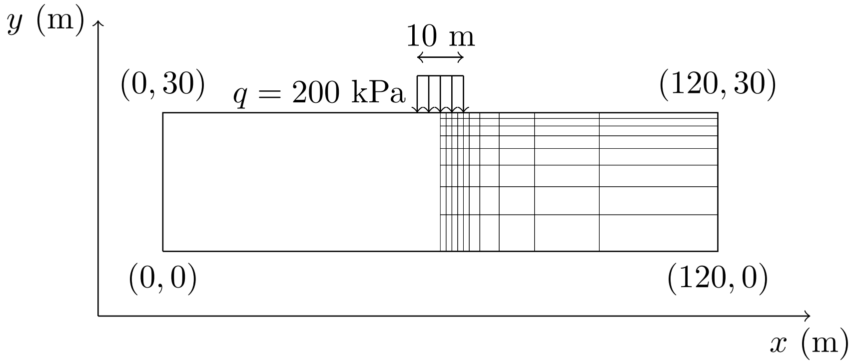
- **Minimising number of FEM analyses.** These are very computationally expensive. For nonlinear FEM, the difference is more pronounced than in linear FEM.
- **Ability to find new test locations quickly**, i.e. transition probabilities not too small.
- **Minimising the number of analyses conducted that do not meet the minimum threshold level.** Don't want to jump too far from current best location. A potential problem for Gibbs sampling, most likely to jump closer to the mean response rather than further away.

Expected behaviour during analysis

Will higher acceptance probability for Gibbs and CMH offset the potentially faster mixing of MH?

- MCS should become less efficient for small probabilities as the convergence rate is fixed
- Componentwise MCMC samplers should mix more slowly - more simulations required to explore sample space
- For computationally expensive sampling (i.e. nonlinear FEM), minimising the number of analyses is the time critical part of analysis
- MH sampler acceptance probability tends to zero for infinite probabilistic vector size. This may reduce simulation efficiency compared to Gibbs and CMH.
- On the other hand, failure to jump to a new location has little computational overhead, FEM equations do not have to be re-evaluated.

Common problem geometry and parameters



Mesh - 80 elements.

Mean(E) $= 50$ MPa, CoV(E) $= 0.2$, $\nu = 0.2$.

Originally from: Sudret & Der Kiureghian (2002)
doi:10.1016/s0266-8920(02)00031-0

Linear and nonlinear analysis

Reliability analysis problem

Find $\mathbb{P}(u > u_0)$. u = vertical displacement at centre of footing.

- Two Subset Simulation tests - linear and nonlinear FEM problem
- Linear problem from: Sudret & Der Kiureghian (2002) doi:10.1016/s0266-8920(02)00031-0
- SSFEM results indicate $\mathbb{P}(u > u_0^i) \approx 10^{-1}\mathbb{P}(u > u_0^{i-1})$
- Nonlinear problem presented adds more complex constitutive model

Limit state values, u_0

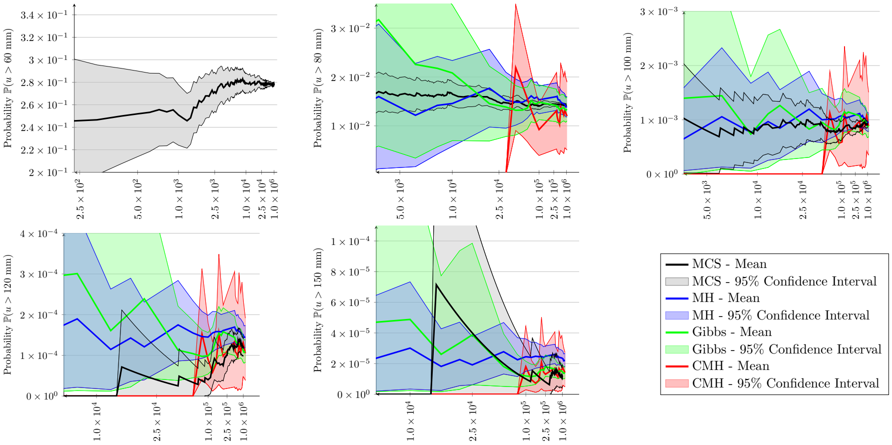
$u_0^1 = 60$ mm
$u_0^2 = 80$ mm
$u_0^3 = 100$ mm
$u_0^4 = 120$ mm
$u_0^5 = 150$ mm

Linear elastic problem

To test performance and convergence, a large number of simulations were carried out.

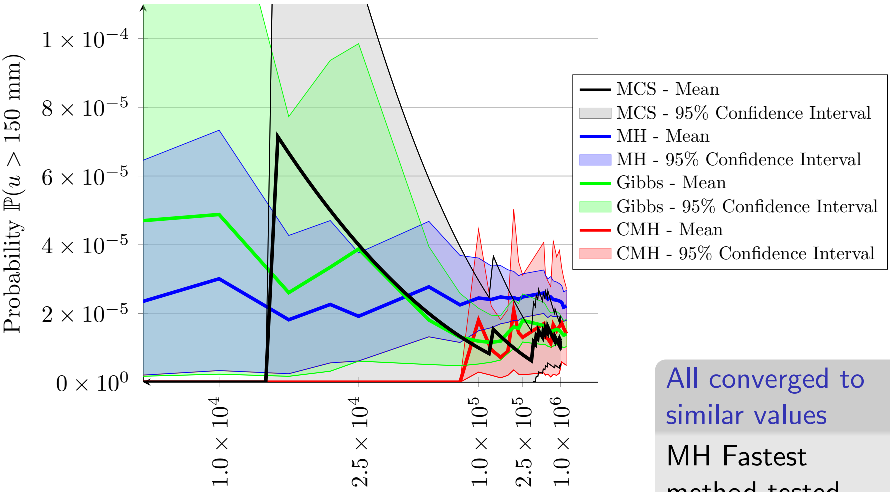
- E random field, $\theta_x = \infty, \theta_y = 30$ m. Normally distributed, exponential decay correlation function.
- MCS, SSFEM and Subset Simulation convergence results compared.
- Subset Simulation - MH, Gibbs and CMH tested.
- MCS - 1×10^6 simulations.
- Subset Simulation - 1×10^5 per level. Number of simulations calculated cumulatively.
- For sampling methodologies: each linear analysis takes same amount of time to solve on average. Number of runs is reasonable for time comparison.

Convergence vs number FEM simulations



X-axis all plots: Number of simulations, N. Scale $= \frac{1}{\sqrt{N}}$

Convergence vs FEM simulations - $\mathbb{P}(u > u_0 = 150 \text{ mm})$

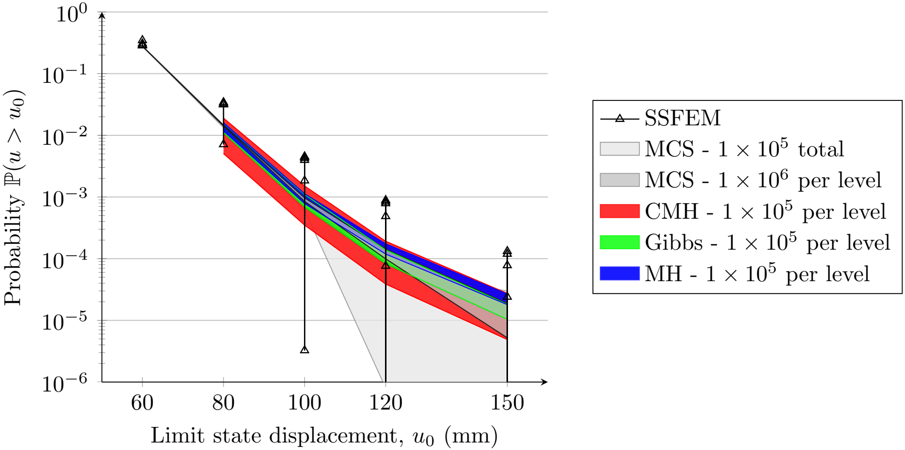


Number of simulations, N. Scale $= \frac{1}{\sqrt{N}}$

All converged to similar values

MH Fastest method tested

Summary results - linear elastic parameters

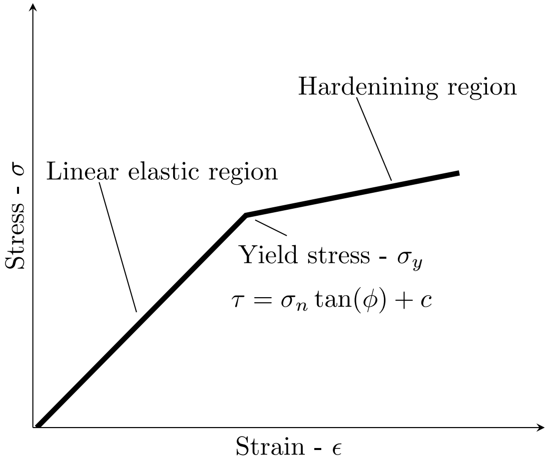


Shaded areas indicate 95% confidence intervals.

Elastic Problem - Discussion

- SSFEM, MCS and Subset Simulation in reasonable agreement. All techniques converged to similar values.
- MCS convergence improved with number of runs as expected. 1×10^6 FEM simulations required to estimate $\mathbb{P}(u > 150 \text{ mm})$ reasonably well.
- Subset Simulation much better than MCS for $u_0 = 120$ and 150 mm.
- MH best convergence performance, followed closely by Gibbs.
- Oscillations in CMH mean prevented confidence interval convergence.
- Confidence interval estimation technique effectively captures range of possible values.

Nonlinear FEM problem - constitutive model



For the nonlinear analysis, a Mohr-Coulomb constitutive model was introduced.

E, ϕ and c - random fields (Normally distributed).

All correlation lengths set to $\theta_x = 10$ m, $\theta_y = 30$ m.

Very small hardening parameter, ψ, included.

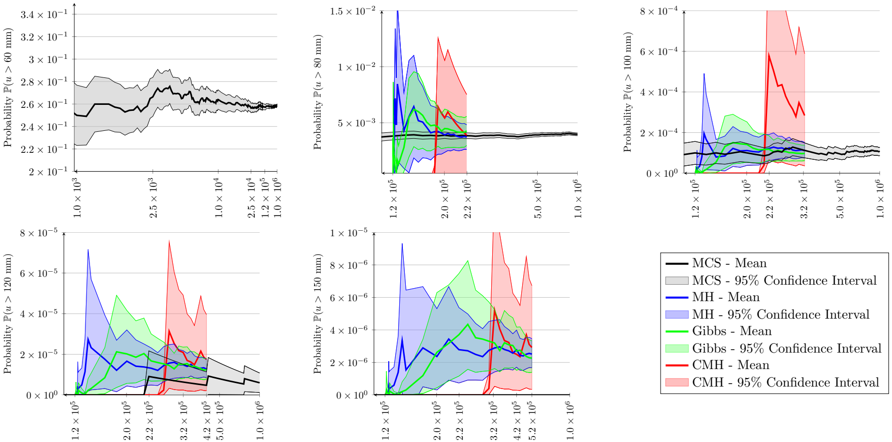
Nonlinear FEM - problem description

To match a more realistic analysis, the following analyses were carried out:

- For $\mathbb{P}(u > u_0 = 60 \text{ mm})$, MCS run until 1% relative error.
- For all other Subset Levels, 1×10^5 MCMC simulations carried out.
- MH, CMH and Gibbs sampling tested.
- To verify, the MCMC analysis 1×10^6 MCS simulations were run.

The convergence rate and efficiency of each method was compared.

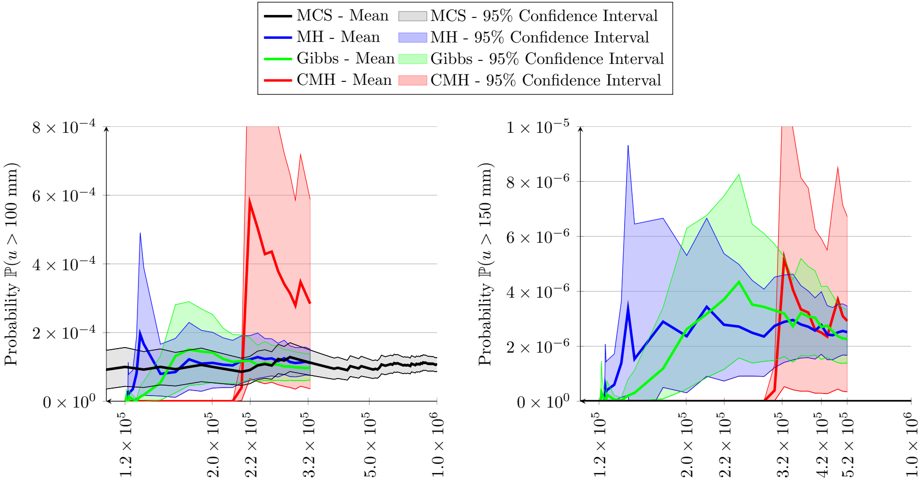
Convergence vs number FEM simulations



X-axis all plots: Number of simulations, N. Scale $= \frac{1}{\sqrt{N}}$

MCS relative error of 1% found after approximately 1.2×10^5 simulations.

CMH - poor estimate for $u_0 = 100$ mm recovery

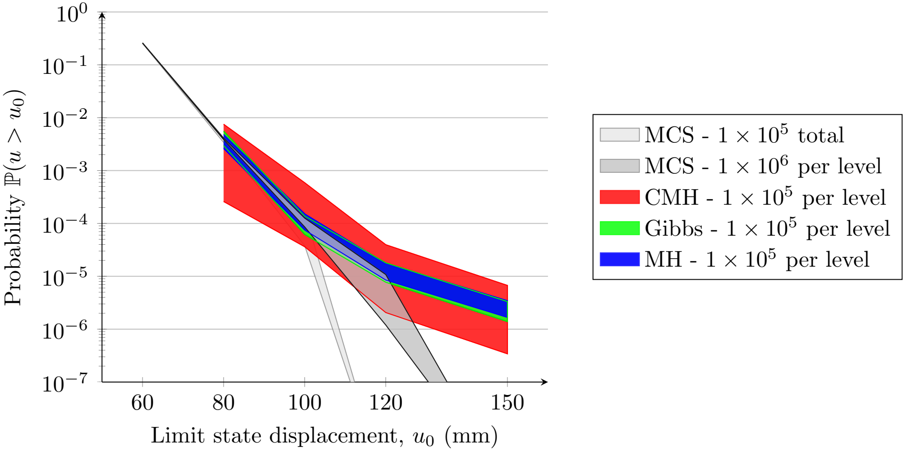


X-axis all plots: Number of simulations, N. Scale $= \frac{1}{\sqrt{N}}$

Nonlinear analysis - trace plots

- As in the elastic case, all values converged to similar estimates.
- For $u_0 = 120$ and 150 mm:
  - $\mathbb{P}(u > 120 \text{ mm}) \approx 10^{-5}$
  - $\mathbb{P}(u > 150 \text{ mm}) \approx 10^{-6}$
- MCS would require millions of runs for accurate estimates for $\mathbb{P}(u > 150 \text{ mm})$ and $u_0 > 150$ mm.
- CMH, again, displayed slower mixing than other MCMC methods. Long period spent below threshold, long period spent above threshold.
- Subset Simulation able to recover from earlier poor estimates.
- Confidence intervals critical for understanding mean estimates.

Summary results - nonlinear parameters



Shaded areas indicate 95% confidence intervals.

Nonlinear analysis - efficiency estimates

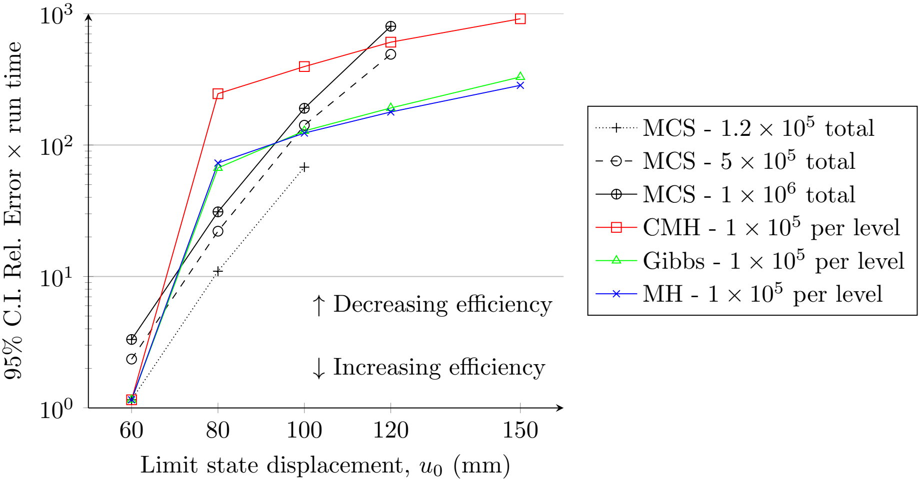
Simulation technique efficiency compared by:

Relative computational cost estimate

$$\text{Computational Cost} = (\text{Run time}) \times (95\% \text{ C.I. Rel. Err.})$$

- Time normalised by setting time for 1×10^5 by MCS simulations to 1 unit
- Relative error taken as average 95% confidence interval width.
- As all Subset Simulation analyses used the same $\mathbb{P}(u > 60 \text{ mm})$, the MCMC efficiency could be compared.

Nonlinear analysis - efficiency



Nonlinear analysis - efficiency

Monte Carlo Simulation:

- More efficient closer to mean.
- By $u_0 = 100$ mm, efficiency becomes worse than Subset Simulation.

Subset Simulation - all methods started using same $\mathbb{P}(u > 60$ mm):

- CMH slow mixing degrades efficiency by preventing confidence interval convergence.
- Gibbs and MH very similar performance.
- Further from mean response, MH begins to outperform Gibbs.
- Analyses suggest that for further from mean responses, MH had best performance.

Nonlinear analysis - conclusions

Sampling efficiency observations and explanations:

- Subset Simulation is more efficient than direct MCS far from the mean.
- Componentwise sampling reduces simulation efficiency.
- Gibbs sampling - away from the mean, sampling from the full marginal makes it more likely to sample below the minimum Subset level threshold (than MH or CMH).
- In contrast, MH sampling doesn't jump far enough to fall below the minimum threshold level as often.
- For computationally expensive sampling, it is better to take more time to find a good sample (by MH) than to take many poor samples (CMH and Gibbs).

High dimensional acceptance probability problem in MH?

If acceptance probability becomes too small, update maximal batches of probabilistic vector per iteration.

Conclusions

- For analyses conducted, Subset Simulation was effective for estimating rare event, far from mean responses.
- Metropolis-Hastings was best performing MCMC sampler.
- Gibbs performance similar, but degraded further from mean.
- Componentwise Metropolis-Hastings displayed oscillatory behaviour around estimated value.
- Confidence interval estimator worked well to capture range of variations.

Thank you!